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# A continuous function space with a Faber basis

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**Abstract**

Let $S \subset \mathbb{R}$ be compact with $\#S = \infty$ and let $C(S)$ be the set of all real continuous functions on $S$. We ask for an algebraic polynomial sequence $(P_n)_{n=0}^{\infty}$ with $\deg P_n = n$ such that every $f \in C(S)$ has a unique representation $f = \sum_{i=0}^{\infty} \alpha_i P_i$ and call such a basis Faber basis. In the special case of $S = S_q = \{q^k; k \in \mathbb{N}_0\} \cup \{0\}$, $0 < q < 1$, we prove the existence of such a basis. A special orthonormal Faber basis is given by the so-called little $q$-Legendre polynomials. Moreover, these polynomials state an example with $A(S_q) \neq U(S_q) = C(S_q)$, where $A(S_q)$ is the so-called Wiener algebra and $U(S_q)$ is the set of all $f \in C(S_q)$ which are uniquely represented by its Fourier series.


*Keywords:* Continuous function; Polynomial; Basis; Little $q$-Legendre polynomials; Fourier series; Wiener algebra

## 1. Introduction and basic facts

Let $S \subset \mathbb{R}$ be compact with $\#S = \infty$ and let $C(S)$ be the set of all real continuous functions on $S$. It is a typical problem to approximate or to represent a function $f \in C(S)$ going back to the set of real algebraic polynomials. In this context there are some important results on approximation. For instance, by the Stone–Weierstrass theorem [1] there exists a real algebraic polynomial $P$ such that $||f - P||_{\infty}$ is arbitrary small. In case of $C([0,1])$ Müntz's theorem [1] is an attractive version of the

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Weierstrass theorem. Another goal is to determine the element of best approximation in $P_{(n)}$, where $P_{(n)}$ denotes the space of all polynomials $P$ with $\deg P \leqslant n$; see [8].

Our aim here is to give a special representation of $f$. For that purpose we refer to the idea of a basis.

**Definition 1.** *A sequence $(f_n)_{n=0}^{\infty}$ in an infinite Banach space $B$ is called basis if for every $f \in B$ there exists a unique sequence of scalars $(\alpha_n)_{n=0}^{\infty}$ such that*

$$f = \sum_{i=0}^{\infty} \alpha_i f_i. \tag{1}$$

In case of $B = C(S)$ a well-known basis is the so-called Schauder basis, see [9], but we are interested in a very special kind of a polynomial basis.

**Definition 2.** *A basis $(P_n)_{n=0}^{\infty}$ of $C(S)$ is called a polynomial basis with strict degrees or Faber basis if $P_n$ is a real algebraic polynomial with* $\deg P_n = n$ *for all $n \in \mathbb{N}_0$.*

There is a famous result of Faber that in case of $S = [a, b]$ there does not exist a polynomial basis with strict degrees; see [3]. The question is, whether there are sets $S$ such that a Faber basis exists. For to investigate this question, the following theorem is very useful.

**Theorem 1.** *The following conditions are equivalent.*

- (i) *There exists a Faber basis $(P_n)_{n=0}^{\infty}$ of $C(S)$.*
- (ii) *There exists a sequence $(v_n)_{n=0}^{\infty}$ of continuous linear operators from $C(S)$* into *$C(S)$ such that*
  - (a) *$v_n(f) \in P_{(n)}$ for all $f \in C(S)$, $n \in \mathbb{N}_0$.*
  - (b) *$v_n(p) = p$ for all $p \in P_{(n)}$, $n \in \mathbb{N}_0$.*
  - (c) *$\lim_{n\to\infty} v_n(f) = f$ for all $f \in C(S)$.*
  - (d) *$\deg v_n(f) \leqslant \deg v_{n+1}(f)$ for all $f \in C(S)$, $n \in \mathbb{N}_0$.*

*If $(Q_n)_{n=0}^{\infty}$ is a sequence of real algebraic polynomials with* $\deg Q_n = n$ *then a Faber basis is given by*

$$P_0 = Q_0, \quad P_n = Q_n - v_{n-1}(Q_n) \quad \textit{for all } n \in \mathbb{N}. \tag{2}$$

For the proof we refer to [9, Theorem 20.1].

Note that according to the Banach–Steinhaus theorem we may replace (c) in Theorem 1 by

$$\|v_n\| < C \quad \text{for all } n \in \mathbb{N}_0. \tag{3}$$

We focus on two special types of a Faber basis.

**Definition 3.**

(i) *A Faber basis* $(l_n)_{n=0}^{\infty}$ *is called Lagrange basis with respect to a sequence of distinct points* $(s_n)_{n=0}^{\infty}$ *in* $S$ *if* $l_n(s_i) = 0$ *for all* $i < n$ *and* $l_n(s_n) = 1$ *for all* $n \in \mathbb{N}_0$.

(ii) *A Faber basis* $(p_n)_{n=0}^{\infty}$ *is called orthonormal basis with respect to a probability measure* $\pi$ *on* $S$ *if* $\int p_n p_m \, d\pi = \delta_{n,m}$ *for all* $n, m \in \mathbb{N}_0$, *where* $\delta_{n,m}$ *denotes Kronecker's delta symbol.*

In case of a Lagrange basis it holds $f = \sum_{i=0}^{\infty} \lambda_i(f) l_i$ with

$$\lambda_i(f) = f(s_i) - \sum_{j=0}^{i-1} \lambda_j(f) l_j(s_i), \tag{4}$$

and in case of an orthonormal basis it holds $f = \sum_{i=0}^{\infty} \mu_i(f) p_i$ with

$$\mu_i(f) = \langle f, p_i \rangle = \int f p_i \, d\pi. \tag{5}$$

Further on we pay particular attention to the set

$$S_q = \{q^k; k \in \mathbb{N}_0\} \cup \{0\}, \quad 0 < q < 1, \tag{6}$$

and prove the existence of a Lagrange basis in Section 2. The set $S_q$ is also well-known as the support of the orthogonality measure which belongs to little $q$-Jacobi polynomials; see [4]. A special case of these polynomials are the so-called little $q$-Legendre polynomials. They have been studied thoroughly and they are relevant to different topics, see for instance [5,10]. Especially, they have positive linearization coefficients, i.e. they are associated with a polynomial hypergroup; see for instance [6]. In Section 3 we prove that little $q$-Legendre polynomials constitute an orthonormal basis of $C(S_q)$.

## 2. Continuous function spaces with a Lagrange basis

In order to obtain spaces $C(S)$ with a Lagrange basis we characterize the situation as follows.

**Lemma 1.** *If* $(l_n)_{n=0}^{\infty}$ *is a Lagrange basis of* $C(S)$ *with respect to a sequence* $(s_n)_{n=0}^{\infty}$, *then* $\{s_0, s_1, \ldots\}$ *is dense in* $S$.

**Proof.** Denote by $X$ the closure of $\{s_0, s_1, \ldots\}$ and assume $x \in S \setminus X$. Then there exist functions $f_1, f_2 \in C(S)$ such that $f_1|_X = f_2|_X$ and $f_1(x) \neq f_2(x)$.

By (4) we get $f_1 = f_2$ which yields a contradiction. □

Due to Lemma 1 one of the most simplest cases to deal with is

$$S = \{s_n; n \in \mathbb{N}_0\} \cup \{s\}, \tag{7}$$

where $s$ is the unique limit point of the sequence $(s_n)_{n=0}^{\infty}$.

Define

$$L_n^i(x) = \frac{\prod_{k=0, k\neq i}^{n}(x - s_k)}{\prod_{k=0, k\neq i}^{n}(s_i - s_k)} \quad \text{for all } n \in \mathbb{N}_0, \quad i = 0, 1, \dots, n. \tag{8}$$

Of course, if there exists a Lagrange basis with respect to $(s_n)_{n=0}^{\infty}$, then it is given by

$$l_n(x) = L_n^n(x) \quad \text{for all } n \in \mathbb{N}_0. \tag{9}$$

**Lemma 2.** *Let $(s_n)_{n=0}^{\infty}$ be a strictly increasing or strictly decreasing sequence with limit point $s$ and $S = \{s_n; n \in \mathbb{N}_0\} \cup \{s\}$.*

*Then there exists a Lagrange basis $(l_n)_{n=0}^{\infty}$ of $C(S)$ with respect to the sequence $(s_n)_{n=0}^{\infty}$ if and only if $\{\sum_{i=0}^{n} |L_n^i(s)|;\ n \in \mathbb{N}_0\}$ is bounded.*

**Proof.** In case of $x \in \{s_0, s_1, \dots, s_n\}$ it holds $\sum_{i=0}^{n} |L_n^i(x)| = 1$ and if $l > m \geqslant n$, then the assumed monotony of the sequence yields $|L_n^i(s_l)| > |L_n^i(s_m)|$ for all $i = 0, 1, \dots, n$.

Hence, $(\sum_{i=0}^{n} |L_n^i(s_k)|)_{k=0}^{\infty}$ is monoton increasing and

$$\max_{x \in S} \sum_{i=0}^{n} |L_n^i(x)| = \sum_{i=0}^{n} |L_n^i(s)|. \tag{10}$$

Define a sequence of continuous linear operators $(v_n)_{n=0}^{\infty}$ from $C(S)$ into $C(S)$ by

$$v_n(f) = \sum_{i=0}^{n} f(s_i) L_n^i, \tag{11}$$

where $v_n(f)$ is the Lagrange interpolation polynomial passing through the points $(s_0, f(s_0)), \dots, (s_n, f(s_n))$.

For the operator norm it holds

$$||v_n|| = \sup_{||f||_\infty \leqslant 1} ||v_n(f)||_\infty \leqslant \max_{x \in S} \sum_{i=0}^{n} |L_n^i(x)|. \tag{12}$$

Choose $g_n \in C(S)$ with $||g_n||_\infty \leqslant 1$ and $g_n(s_i) = \operatorname{sign} L_n^i(s)$. Hence, $||v_n(g_n)||_\infty = \sum_{i=0}^{n} |L_n^i(s)|$ and

$$||v_n|| = \sum_{i=0}^{n} |L_n^i(s)|. \tag{13}$$

For the rest of the proof we refer to Theorem 1 and

$$\sum_{i=0}^{n} f(s_i)L_n^i = \sum_{i=0}^{n} \lambda_i(f)l_i. \quad \square \tag{14}$$

Now, we are able to prove the following theorem in case of $S = S_q$.

**Theorem 2.** *In case of $C(S_q)$ there exists a Lagrange basis $(l_n)_{n=0}^{\infty}$ with respect to the sequence $s_n = q^n$, $n \in \mathbb{N}_0$.*

**Proof.** It is easy to check that

$$|L_n^0(0)| = \frac{q^{n(n+1)/2}}{\prod_{k=1}^{n}(1-q^k)} \quad \text{for all } n \in \mathbb{N}_0, \tag{15}$$

and

$$|L_{n+1}^i(0)| = \frac{1}{1-q^i}|L_n^{i-1}(0)| \quad \text{for all } n \in \mathbb{N}_0, \quad i = 1, 2, \ldots, n. \tag{16}$$

Therefore,

$$\begin{aligned} \sum_{i=0}^{n} |L_n^i(0)| &= \sum_{i=0}^{n} \prod_{j=1}^{n-i} \frac{1}{1-q^j} |L_i^0(0)| \\ &\leqslant \prod_{j=1}^{\infty} \frac{1}{1-q^j} \sum_{i=0}^{n} |L_i^0(0)| \\ &\leqslant \prod_{j=1}^{\infty} \frac{1}{1-q^j} \sum_{i=0}^{\infty} \frac{q^{i(i+1)/2}}{\prod_{k=1}^{i}(1-q^k)}. \end{aligned} \tag{17}$$

The product and the series on the right-hand side are finite by standard arguments and independent from $n$. Now, by Lemma 2 the proof is complete. $\square$

This is not true for an arbitrary set $S$ of shape (7). In order to give a counter-example let

$$S^r = \{(k+1)^{-r}; k \in \mathbb{N}_0\} \cup \{0\}, \quad 0 < r < \infty, \tag{18}$$

and $s_n = (n+1)^{-r}$, $n \in \mathbb{N}_0$. By simple calculations we obtain

$$L_n^n(0) = \frac{1}{\prod_{i=1}^{n}(1 - (\frac{i}{n+1})^r)} \tag{19}$$

and $\lim_{n\to\infty} L_n^n(0) = \infty$. Hence, by Lemma 2 there is no Lagrange basis of $C(S^r)$ with respect to the sequence $(s_n)_{n=0}^{\infty}$.

In the next section we give a special orthonormal basis of $C(S_q)$.

## 3. Little $q$-Legendre polynomials

Let us define a probability measure $\pi$ on $S_q$ by

$$\pi(q^k) = q^k(1-q) \quad \text{for all } k \in \mathbb{N}_0, \quad \text{and} \quad \pi(0) = 0. \tag{20}$$

The orthogonal polynomials $(R_n)_{n=0}^{\infty}$ with respect to $\pi$ are called little $q$-Legendre polynomials. They satisfy a three term recurrence relation

$$R_1(x)R_n(x) = a_n R_{n+1}(x) + b_n R_n(x) + c_n R_{n-1}(x), \quad n \geqslant 1, \tag{21}$$

with $R_0(x) = 1$ and $R_1(x) = 1 - (q+1)x$, where

$$a_n = q^n \frac{(1+q)(1-q^{n+1})}{(1-q^{2n+1})(1+q^{n+1})}, \tag{22}$$

$$b_n = \frac{(1-q^n)(1-q^{n+1})}{(1+q^n)(1+q^{n+1})}, \tag{23}$$

$$c_n = q^n \frac{(1+q)(1-q^n)}{(1-q^{2n+1})(1+q^n)}. \tag{24}$$

It holds the orthogonality relation

$$\sum_{k=0}^{\infty} q^k(1-q)R_n(q^k)R_m(q^k) = \frac{(1-q)q^n}{1-q^{2n+1}}\,\delta_{n,m}, \tag{25}$$

see [4]. The little $q$-Legendre polynomials are normalized by

$$R_n(0) = 1 \quad \text{for all } n \in \mathbb{N}_0, \tag{26}$$

and they are associated with a so-called hypergroup structure on $\mathbb{N}_0$; see [7]. Therefore, it follows

$$\max_{\xi \in S_q} |R_n(\xi)| = R_n(0) = 1 \quad \text{for all } n \in \mathbb{N}_0. \tag{27}$$

The orthonormal little $q$-Legendre polynomials are defined by

$$p_n = \sqrt{\frac{1-q^{2n+1}}{(1-q)q^n}}\, R_n, \tag{28}$$

and we set

$$h(n) = (p_n(0))^2 = \frac{1-q^{2n+1}}{(1-q)q^n} \quad \text{for all} \quad n \in \mathbb{N}_0. \tag{29}$$

For $x \neq y$ we obtain by Christoffel–Darboux formula [2]

$$\sum_{k=0}^{n} R_k(x)R_k(y)h(k) = \frac{a_n h(n)}{q+1} \frac{R_n(x)R_{n+1}(y) - R_{n+1}(x)R_n(y)}{x-y}, \quad n \in \mathbb{N}_0. \tag{30}$$

Now, we are able to prove the following result.

**Theorem 3.** *The sequence of orthonormal little q-Legendre polynomials* $(p_n)_{n=0}^{\infty}$ *is a Faber basis of* $C(S_q)$.

**Proof.** For $n \in \mathbb{N}_0$ define a continuous linear transformation $v_n$ from $C(S)$ into $C(S)$ by

$$v_n(f) = \sum_{i=0}^{n} \langle f, p_i \rangle p_i. \tag{31}$$

By Theorem 1 and (3) it remains to prove that there exists a real number $C > 0$ with

$$||v_n|| = \sup_{||f||_\infty \leqslant 1} ||v_n(f)||_\infty \leqslant C \quad \text{for all } n \in \mathbb{N}_0. \tag{32}$$

For arbitrary $x \in S_q, f \in C(S_q)$ with $||f||_\infty \leqslant 1$, we have

$$\begin{aligned} v_n(f)(x) &= \sum_{i=0}^{n} \sum_{j=0}^{\infty} f(q^j)p_i(q^j)q^j(1-q)p_i(x) \\ &= \sum_{i=0}^{n} \sum_{j=0}^{n} f(q^j)p_i(q^j)q^j(1-q)p_i(x) \\ &\quad + \sum_{i=0}^{n} \sum_{j=n+1}^{\infty} f(q^j)p_i(q^j)q^j(1-q)p_i(x) \\ &= S1_n(f,x) + S2_n(f,x). \end{aligned} \tag{33}$$

By (27) and (29) it follows

$$\begin{aligned} |S2_n(f,x)| &= (1-q)\left| \sum_{i=0}^{n} \sum_{j=n+1}^{\infty} f(q^j)p_i(q^j)q^j p_i(x) \right| \\ &\leqslant (1-q) \sum_{i=0}^{n} (p_i(0))^2 \sum_{j=n+1}^{\infty} q^j \end{aligned}$$

$$= \sum_{i=0}^{n} (p_i(0))^2 q^{n+1} = \sum_{i=0}^{n} \frac{1-q^{2i+1}}{(1-q)q^i} q^{n+1}$$

$$\leqslant \frac{1}{1-q} \sum_{i=0}^{n} \frac{q^{n+1}}{q^i} \leqslant \frac{1}{(1-q)^2} \quad \text{for all } n \in \mathbb{N}_0. \tag{34}$$

Next, we give an upper bound for $S1_n(f,x)$ which is independent of $n$. Replacing $x$ by $q^m$, $m \in \mathbb{N}_0 \cup \{\infty\}$, where $q^\infty = 0$, we get

$$\begin{aligned} |S1_n(f,x)| &= (1-q)\left| \sum_{i=0}^{n} \sum_{j=0}^{n} f(q^j) p_i(q^j) q^j p_i(q^m) \right| \\ &\leqslant (1-q) \sum_{j=0}^{n} q^j \left| \sum_{i=0}^{n} p_i(q^j) p_i(q^m) \right| \\ &\leqslant (1-q) \sum_{j=0, j\neq m}^{n} q^j \left| \sum_{i=0}^{n} p_i(q^j) p_i(q^m) \right| \\ &\quad + (1-q) q^m \sum_{i=0}^{n} (p_i(q^m))^2. \end{aligned} \tag{35}$$

Since $j \neq m$, we obtain by Christoffel–Darboux formula (30) and (27)

$$\begin{aligned} \left| \sum_{i=0}^{n} p_i(q^j) p_i(q^m) \right| &= \frac{1-q^{n+1}}{1+q^{n+1}} \frac{1}{1-q} \frac{|R_{n+1}(q^j) R_n(q^m) - R_n(q^j) R_{n+1}(q^m)|}{|q^j - q^m|} \\ &\leqslant \frac{1}{1-q} \frac{|R_{n+1}(q^j)| + |R_n(q^j)|}{|q^j - q^m|}. \end{aligned} \tag{36}$$

Hence,

$$\begin{aligned} &(1-q) \sum_{j=0, j\neq m}^{n} q^j \left| \sum_{i=0}^{n} p_i(q^j) p_i(q^m) \right| \\ &\quad \leqslant \sum_{j=0, j\neq m}^{n} \frac{q^j}{|q^j - q^m|} (|R_{n+1}(q^j)| + |R_n(q^j)|) \\ &\quad \leqslant \frac{1}{1-q} \left( \sum_{j=0}^{n+1} |R_{n+1}(q^j)| + \sum_{j=0}^{n} |R_n(q^j)| \right). \end{aligned} \tag{37}$$

By Cauchy–Schwarz inequality we derive

$$\begin{aligned}\sum_{j=0}^{n}|R_n(q^j)| &= \frac{1}{1-q}\sum_{j=0}^{n}(1-q)q^j\frac{1}{q^j\sqrt{h(n)}}|p_n(q^j)|\\ &\leqslant \frac{1}{1-q}\sqrt{\sum_{j=0}^{n}(1-q)q^j\frac{1}{q^{2j}h(n)}}\sqrt{\sum_{j=0}^{n}(1-q)q^j(p_n(q^j))^2}\\ &\leqslant \frac{1}{1-q}\sqrt{(1-q)\sum_{j=0}^{n}q^{n-j}\frac{1-q}{1-q^{2n+1}}}\\ &\leqslant \frac{1}{1-q}\sqrt{(1-q)\sum_{j=0}^{n}q^{n-j}}\leqslant\frac{1}{1-q}.\end{aligned} \tag{38}$$

In case of $m=\infty$ the second sum in (35) equals 0. Otherwise, it holds $\pi(q^m)>0$ and

$$\sum_{i=0}^{n}(p_i(q^m))^2\leqslant\frac{1}{\pi(q^m)}=\frac{1}{(1-q)q^m}, \tag{39}$$

see [2]. To summarize, we have shown

$$|S1_n(f,x)|\leqslant\frac{2}{(1-q)^2}+1 \quad \text{for all } n\in\mathbb{N}_0. \tag{40}$$

Finally, with $C=\frac{4-2q+q^2}{(1-q)^2}$ the proof is complete. □

One crucial point within the proof of Theorem 3 was to make use of (27) which holds in case of little $q$-Legendre polynomials but does not hold in general.

For the polynomial hypergroup which is associated with the sequence $(R_n)_{n=0}^{\infty}$ the so-called Wiener algebra $A(S_q)$, see [7], is defined by

$$A(S_q)=\{f\in C(S_q):\hat{f}\in l^1(\mathbb{N}_0,h)\}, \tag{41}$$

where

$$\hat{f}(k)=\int fR_k\,d\pi \quad \text{for all } k\in\mathbb{N}_0. \tag{42}$$

Of course, $A(S_q)\subset U(S_q)$, where $U(S_q)$ denotes the set of all functions $f\in C(S_q)$ which are uniquely represented by its Fourier series

$$f=\sum_{k=0}^{\infty}\hat{f}(k)R_kh(k). \tag{43}$$

In [7] we have proven that $A(S_q) \neq C(S_q)$. Now, by Theorem 3 we have shown that $U(S_q) = C(S_q)$, and therefore,

$$A(S_q) \neq U(S_q) = C(S_q). \tag{44}$$

We should mention that due to Theorem 3 and former results, see [6], the little $q$-Legendre polynomials also constitute a basis of the Banach spaces $L^p(S_q, \pi)$, $1 \leqslant p < \infty$.

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